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String theory on lorentzian AdS_3 in minisuperspace

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Abstract: We investigate string theory on lorentzian AdS_3 in the minisuperspace approximation. The minisuperspace model reduces to the worldline theory of a scalar particle in the lorentzian AdS_3. The Hilbert space consists of normalizable wave functions, and we see that the unitarity of the theory (or the self-adjointness of the hamiltonian) restricts the possible sets of wave functions. The restricted wave functions have the property of probability conservation (or current conservation) across the horizons. Two and three point functions are also computed. In the euclidean model functional forms of these quantities are restricted by the $\mathrm{SL}(2,\mathbb{R})$ symmetry almost uniquely, however, in the lorentzian model there are several ambiguities left. The ambiguities are fixed by the direct computation of overlaps of wave functions.

Keywords: Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

Superstring theory on Anti-de Sitter (AdS) space attracts many attentions recently due to the AdS/CFT correspondence [1]–[3]. According to the AdS/CFT correspondence, superstring theory on $(d+1)$-dimensional Anti-de Sitter space is dual to d-dimensional conformal field theory (CFT) defined at the boundary of AdS. This implies that correlators in AdS and those in CFT are mappable each other. However, superstring theory on AdS is difficult to deal with, because we do not know how to quantize superstrings on a target space including RR-flux in general (except for the pp-wave case [4, 5]).

The only tractable case is superstrings on AdS_3 with NSNS-flux, which is dual to 2-dimensional CFT at the boundary [6]–[8]. Superstring theory on euclidean AdS_3 can be described by $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ Wess-Zumino-Novikov-Witten (WZNW) model. The WZNW model was investigated in detail, and in particular the minisuperspace limit (the limit where we neglect the dependence of the spatial coordinate σ) was discussed in [9, 10], and three point function was obtained exactly in [11]. Superstring theory on *lorentzian* AdS_3 can be described by $\mathrm{SL}(2,\mathbb{R})$ WZNW model, however this model is much more difficult than its euclidean counterpart (the difficulty was discussed, for example, in [12]). In particular, the precise spectrum was discovered only recently [13, 14]. Moreover, correlation functions in the $\mathrm{SL}(2,\mathbb{R})$ WZNW theory were conjectured to be given by the analytic continuation of those in $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZNW model, and there is no direct derivation of them (see, for the previous discussion, [15]–[17]).

In this paper, we would like to deal with the lorentzian model directly although in the minisuperspace limit. As it stands now, complete definition of string theory on a curved lorentzian spacetime is not known. The only proposal is to utilize the analytic continuation of the euclidean counterpart. In the minisuperspace limit, one can do better as the theory is reduced to a quantum mechanical system, and hence can treat the theory directly in the lorentzian signature. The Hilbert space is constructed by square integrable wave functions, however the wave functions are not compatible to the unitarity of the theory in general. We follow a general theory of self-adjointness of a linear operator (for a review, see [18]) to construct the domain of self-adjoint hamiltonian which includes a set of restricted wave functions.

A situation analogous to this is timelike Liouville theory [19, 20]. Timelike Liouville theory arises as a continuum worldsheet description of spacelike S-brane [21] and rolling tachyon [22]. In [19, 20] and [23]–[25], timelike Liouville theory was defined as the analytic continuation of spacelike Liouville theory, and correlation functions are calculated. Recently, in [26], utilizing the reduced model of $\mathrm{SL}(2,\mathbb{R})$ WZNW model [27, 28], the minisuperspace limit of timelike Liouville theory was investigated directly, and the minisuperspace model was compared with the analytic continued one.

This paper is organized as follows. In section 2, we investigate normalizable states of the minisuperspace AdS_3 string theory. The self-adjointness condition is equivalent to the boundary condition of wave functions, and we construct a one parameter family of self-adjoint extensions of the hamiltonian. In the global coordinates of lorentzian AdS, the normalizable wave functions are unique as in [30], and there is unique self-adjoint extension. However, in the Poincaré coordinates adopted in the context, the normalizable wave functions are not unique, and the self-adjointness condition restricts the wave functions to those satisfying the boundary condition. There are horizons in the Poincaré coordinates, and the boundary condition means the probability conservation (or the current conservation) across the horizons.

In section 3, we construct primary fields and compute correlation functions. In order to construct primary fields, we introduce parameters $(x, \bar{x})$, which may be interpreted as coordinates of the boundary of AdS_3 in the sense of AdS/CFT correspondence [7, 8]. The advantage of introducing the parameters is that the transformation of the $\mathrm{SL}(2,\mathbb{R})$ symmetry can be generated by differential operators, and primary fields are constructed as solutions to differential equations. We can find that the primary field is precisely the Fourier transform of wave function constructed in section 2. The solution is not unique and

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it corresponds to the non-uniqueness of the wave functions. Two or three point functions are also obtained as solutions to differential equations. The solutions are almost unique in the euclidean theory, however the solutions in the lorentzian theory have more ambiguities undetermined by the $\mathrm{SL}(2,\mathbb{R})$ symmetry. In the minisuperspace limit, we can compute the correlation functions as overlaps of the wave functions and fix the ambiguities, but in the full CFT case, we may have to use the symmetry to compute the correlation functions as in the case of the euclidean theory [11].

Section 4 is devoted to conclusion and discussions. In appendix A, we summarize various formulae relevant for computations in this paper.

2. Spectrum of the minisuperspace model

String theory on the euclidean AdS_3, which is known to be described by $\mathrm{SL}(2;\mathbb{C})/\mathrm{SU}(2)$ WZNW model, has been investigated for a decade, for example, in [9]–[11]. The theory is an example of non-rational conformal field theory, and in general non-rational conformal field theory is difficult to analyze because it has infinitely many primary fields; only spacelike Liouville theory and $\mathrm{SL}(2;\mathbb{C})/\mathrm{SU}(2)$ WZNW model were solved. The truncation in the minisuperspace limit (in case of spacelike Liouville theory, see [31]) gives a theory only with zero-mode subspace, however it still includes infinite dimensional primary fields. The truncations in spacelike Liouville theory and $\mathrm{SL}(2;\mathbb{C})/\mathrm{SU}(2)$ WZNW model played important roles on the investigation of full theories, so we could expect that the minisuperspace limit of $\mathrm{SL}(2;\mathbb{R})$ WZNW model gives insights in the understanding of the full CFT model. In this section, we will investigate the spectrum by canonical quantization of the minisuperspace model.

2.1 AdS_3 space and $\mathrm{SL}(2;\mathbb{R})$ group elements

The lorentzian AdS_3 space is defined as a hypersurface

$$-X_0^2 + X_1^2 + X_2^2 - X_3^2 = -L_{AdS}^2 \,, \tag{2.1}$$

in $(2+2)$-dimensional embedding flat space $\mathbb{R}^{2,2}$ of signature $(-,+,+,-)$. It is convenient to represent the hypersurface in terms of matrix g:

$$g = \frac{1}{L_{AdS}} \begin{pmatrix} X_1 + X_3 & X_0 - X_2 \\ -X_0 - X_2 & -X_1 + X_3 \end{pmatrix} \qquad \text{where} \qquad \det g = 1 \,, \tag{2.2}$$

viz. group elements of $\mathrm{SL}(2,\mathbb{R})$. The isometry $\mathrm{SO}(2,2) \simeq \mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ on the lorentzian AdS_3 is then realized as left and right action on the group element g. The conformal field theory whose target space is the $\mathrm{SL}(2,\mathbb{R})$ group manifold, viz. the $\mathrm{SL}(2,\mathbb{R})$ WZNW model, then describes string propagation on the lorentzian AdS_3.

One useful parametrization of the group elements is

$$g = e^{i\theta_L \sigma_2} e^{\rho\sigma_3} e^{i\theta_R \sigma_2} \,, \qquad \theta_L = \frac{1}{2}(t+\varphi) \,, \qquad \theta_R = \frac{1}{2}(t-\varphi) \,, \tag{2.3}$$

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where σ_i $(i = 1, 2, 3)$ represents the Pauli matrices. These parameters correspond to the global coordinates of AdS_3, and the metric is written as

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\varphi^2 \,. \tag{2.4}$$

The parameters ρ and φ run $-\infty < \rho < \infty$ and $0 \leq \varphi < 2\pi$, respectively. The boundaries of AdS_3 at $\rho = \pm\infty$ are connected to each other, so there is only one boundary. For the single cover of AdS_3 space, the time direction is periodic $0 \leq t < 2\pi$ and covers the whole spacetime once. For the universal cover of AdS_3 space, we unwrap the closed timelike curve $-\infty < t < \infty$. We will consider only the universal cover of AdS_3 without mention. The euclidean AdS_3 can be obtained from the lorentzian AdS_3 by the Wick rotation $X_0 \to iX_E$ or equivalently $t \to it$.

There is another useful parametrization of the group elements g, based on the Gauss decomposition:

$$g = \begin{pmatrix} 1 & \bar\gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} . \tag{2.5}$$

In this parametrization, the lorentzian AdS_3 is described by so-called Poincaré coordinates

$$ds^2 = \frac{1}{z^2} \left[dz^2 + d\gamma d\bar\gamma \right] . \tag{2.6}$$

There are two patches for the single cover of AdS_3 space, and the coordinate z ranges over $(-\infty, -0)$ and $(+0, +\infty)$ for the each patch. The other coordinates $\gamma, \bar\gamma$ range over $-\infty < \gamma, \bar\gamma < \infty$; they are independent and should not be thought as complex conjugates each other. The patch with $z > 0$ covers half of the spacetime $-X_1 + X_3 > 0$ and there is a boundary at $z = +0$ and a horizon at $z = +\infty$. The other half of the spacetime $-X_1 + X_3 < 0$ is obtained by replacing $z \leftrightarrow -z$, and the patch is glued at the horizon $z = -\infty$ with the other patch. In order to cover whole the universal cover of AdS_3, we need infinitely many Poincaré patches and glue the patches with $z > 0$ and $z < 0$ alternately. If we only deal with a patch, then we often adopt the coordinate $z = \pm e^{-\phi}$ $-\infty < \phi < +\infty$ with the metric

$$ds^2 = d\phi^2 + e^{2\phi} d\gamma d\bar\gamma \,. \tag{2.7}$$

Note that the euclidean AdS_3 is obtainable by the Wick rotation: $X_0 \to iX_E$, viz. $t' \to it'$ in the parametrization of $\gamma = 1/2(\theta + t')$ and $\bar\gamma = 1/2(\theta - t')$. Then, in the euclidean AdS_3, γ and $\bar\gamma$ are complex conjugates each other.

2.2 Minisuperspace limit of SL(2; $\mathbb{R}$) WZNW model

String dynamics on lorentzian AdS_3 is described by the SL(2, $\mathbb{R}$) WZNW model action

$$S = \frac{k}{4\pi} \int_\Sigma d\tau d\sigma \, \mathrm{Tr}[\partial g^{-1} \bar\partial g] + k\Gamma_{WZ} \,, \tag{2.8}$$

where Γ_{WZ} refers to the Wess-Zumino term. We take the lorentzian worldsheet of topology $\mathbb{R} \times \mathbb{S}^1$, and denote derivatives as $\partial = 1/2(\partial_\tau + \partial_\sigma)$ and $\bar\partial = 1/2(\partial_\tau - \partial_\sigma)$. As a parametrization of the group elements, we will adopt the Gauss decomposition (2.5) even though we need infinitely many patches. As we will see below, the minisuperspace model reduces to a quantum mechanics, and the states can be represented as wave functions. The form of wave functions is simpler in the Poincaré coordinates than in the global coordinates.[1] The cost of this is that we have to take care of the connection between each two adjacent patches. Of course, it is just a technical problem and the physics must be the same in the both coordinate systems.

In the chosen Gauss decomposition (2.5), string dynamics on lorentzian AdS_3 is described by the SL$(2,\mathbb{R})$ WZNW model action

$$S = \frac{k}{2\pi}\int_\Sigma d\tau d\sigma \frac{1}{2z^2}[2\partial z\bar{\partial}z + \partial\bar{\gamma}\bar{\partial}\gamma + \partial\gamma\bar{\partial}\bar{\gamma}] + k\Gamma_{WZ}\,. \tag{2.9}$$

As mentioned above, there are two types of patches, and one of them has the coordinate $z > 0$ and the other has $z < 0$. We shall adopt $\arg z = \pi$ in the latter type of patch for definiteness. In the minisuperspace limit, the string is treated as a rigid body, so the worldsheet fields $(z, \gamma, \bar{\gamma})$ become independent of the $\mathbb{S}^1$ coordinate σ. In this limit, the Wess-Zumino term drops out automatically, and the action is reduced to

$$S = \frac{k}{4}\int d\tau \frac{1}{z^2}[(\partial_\tau z)^2 + \partial_\tau\bar{\gamma}\partial_\tau\gamma] = \frac{k}{4}\int d\tau[(\partial_\tau\phi)^2 + e^{2\phi}\partial_\tau\overline{\gamma}\partial_\tau\gamma]\,. \tag{2.10}$$

The classical hamiltonian can be calculated as[2]

$$H_{ws} = (p_\phi)^2 + 4e^{-2\phi}p_\gamma p_{\bar{\gamma}}\,. \tag{2.11}$$

In general, for a given classical hamiltonian, corresponding quantum hamiltonian is afflicted by operator-ordering ambiguity. Upon quantization, the canonical momentum p_ϕ conjugate to the radial coordinate ϕ is promoted to $p_\phi = z\partial_z$, so the quantum hamiltonian would take one of the following forms:

$$H_{ws} = z^a\frac{\partial}{\partial z}z^b\frac{\partial}{\partial z}z^c + 4z^2\frac{\partial}{\partial\gamma}\frac{\partial}{\partial\bar{\gamma}} \qquad \text{where} \qquad a+b+c = 2\,. \tag{2.12}$$

A direct way of prescribing the quantum hamiltonian is by taking the invariant Lichnerowicz operator on SL$(2;\mathbb{R})$ group manifold:

$$H_{ws} := \Box_{\rm AdS} \equiv \frac{1}{\sqrt{-\det g}}\partial_m\left(\sqrt{-\det g}g^{mn}\partial_n\right)\,. \tag{2.13}$$

It is readily seen that, in this prescription, the momentum operators are Weyl-ordered. In fact, the eigenfunctions of the quantum hamiltonian can be interpreted as wave functions satisfying a Klein-Gordon equation in the lorentzian AdS_3

$$\Box_{\rm AdS}\Phi(z,\gamma,\bar{\gamma}) = 4j(j-1)\Phi(z,\gamma,\bar{\gamma})\,, \qquad \Box_{\rm AdS} \equiv z^2\frac{\partial^2}{\partial z^2} - z\frac{\partial}{\partial z} + 4z^2\frac{\partial}{\partial\gamma}\frac{\partial}{\partial\bar{\gamma}}\,. \tag{2.14}$$

Here the mass square is written in terms of the Casimir invariant of the SL$(2,\mathbb{R})$ Lie algebra ($c_2 = j(j-1)$) for later convenience.

[1] In the global coordinates, the wave functions are expressed by hypergeometric functions. Later we compute overlaps of the wave functions, and it is more complicated to perform the integrals of three hypergeometric functions.

[2] We rescale the overall factor $k \to 1$ because it affects only the overall normalization.

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2.3 Normalizable wave functions

The wave function satisfying the Klein-Gordon equation (2.14), or more precisely, the eigenfunction of the quantum hamiltonian, is given by reduction of the phase-space:

$$\Phi(z,\gamma,\bar{\gamma}) = e^{i\lambda\gamma+i\mu\bar{\gamma}} z\, U(z)\,, \tag{2.15}$$

where $U(z)$ is the reduced wave function, referred as Liouville wave function, obeying the zero-energy Schrödinger equation:

$$\left(z^2\frac{\partial^2}{\partial z^2} + z\frac{\partial}{\partial z} - V(z)\right) U(z) = 0 \qquad \text{where} \qquad V(z) = 4z^2\lambda\mu - (2j-1)^2\,. \tag{2.16}$$

The Liouville wave function $U(z)$ is solved in general by a linear combination of the Bessel functions. Depending on the reduction branches, two distinct behaviors are expected. For $\lambda\mu > 0$, the 'potential' $V(z)$ is bounded from below. For $\lambda\mu < 0$, the potential is not bounded from below, so a care should be exercised in this case by prescribing carefully behavior of the wave functions at the "boundary".

We will look for the solutions to (2.16) which are square normalizable (including delta functional normalizable) with respect to the inner product

$$\langle\Phi_2,\Phi_1\rangle \equiv \int_{\mathrm{SL}(2,\mathbb{R})} dg\, \Phi_2^*(g)\Phi_1(g)\,, \tag{2.17}$$

where $dg \equiv d\gamma d\bar{\gamma} dz |z|^{-3}$ is the $\mathrm{SL}(2,\mathbb{R})$-invariant measure. As mentioned above, there are infinitely many pairs of two adjacent patches; one patch has the coordinate $z > 0$ and the other has $z < 0$. We assume that the wave functions on the each pair of patches are the same, and for this reason we will pick up one of the pairs.[3] Then, the inner product we will use is written in

$$\begin{aligned} \langle\Phi_2,\Phi_1\rangle &= \langle\Phi_2,\Phi_1\rangle_- + \langle\Phi_2,\Phi_1\rangle_+ \\ &= \int_{-\infty}^{\infty} d\gamma \int_{-\infty}^{\infty} d\bar{\gamma} \int_{-\infty}^{-0} \frac{dz}{|z|^3}\Phi_2^*(g)\Phi_1(g) + \int_{-\infty}^{\infty} d\gamma \int_{-\infty}^{\infty} d\bar{\gamma} \int_{+0}^{\infty} \frac{dz}{|z|^3}\Phi_2^*(g)\Phi_1(g)\,. \end{aligned} \tag{2.18}$$

In the following, the square integrability is examined with respect to the above inner product.

For $z > 0$, the square integrable solutions to the Klein-Gordon equation (2.16) with real valued eigenvalues $4j(j-1) \in \mathbb{R}$ are given by

$$a_1 K_{2j-1}\left(2\sqrt{\lambda\mu}z\right) \qquad \left(\lambda\mu > 0, j = \frac{1}{2} + i\mathbb{R}\right), \tag{2.19}$$

$$a_2 J_{2j-1}\left(2\sqrt{-\lambda\mu}z\right) \qquad \left(\lambda\mu < 0, j > \frac{1}{2}\right), \tag{2.20}$$

$$a_3 J_{2j-1}\left(2\sqrt{-\lambda\mu}z\right) + a_4 J_{1-2j}\left(2\sqrt{-\lambda\mu}z\right) \qquad \left(\lambda\mu < 0, j = \frac{1}{2} + i\mathbb{R}\right), \tag{2.21}$$

[3] This might be justified by the fact that physics in the global coordinates and in the Poincaré coordinates must be same. There is time-translation invariance in the global coordinates, therefore it is natural to assume that wave functions in next pair of Poincaré patches are the same as those in the original pair.

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with z-independent constants $a_i \in \mathbb{C},\ i = 1, 2, 3, 4$ (which may have dependence on j). We denoted $J_{2j-1}(x)$ as the Bessel functions of the first kind and $K_{2j-1}(x)$ as the modified Bessel function of the second kind, respectively. Depending on the sign of $\lambda\mu$, the potential $V(z)$ in (2.16) pushes the wave function either to $z = 0$ or ∞, so an appropriate 'boundary' condition needs to be prescribed at $z = 0, \infty$ so that the hamiltonian (2.13) maintains self-adjointness.[4] For $z > 0$, the solutions are

$$b_1 K_{2j-1}\left(2\sqrt{\lambda\mu z}\right) \qquad \left(\lambda\mu > 0, j = \frac{1}{2} + i\mathbb{R}\right), \tag{2.22}$$

$$b_2 J_{2j-1}\left(2\sqrt{-\lambda\mu z}\right) \qquad \left(\lambda\mu < 0, j > \frac{1}{2}\right), \tag{2.23}$$

$$b_3 J_{2j-1}\left(2\sqrt{-\lambda\mu z}\right) + b_4 J_{1-2j}\left(2\sqrt{-\lambda\mu z}\right) \qquad \left(\lambda\mu < 0, j = \frac{1}{2} + i\mathbb{R}\right), \tag{2.24}$$

with z-independent constants $b_i \in \mathbb{C},\ i = 1, 2, 3, 4$. If there is no interaction between patches, then we can choose the coefficients b_i independent to a_i. Otherwise, b_i depend on a_i.

2.4 Self-adjointness of the hamiltonian

In examining self-adjointness of the hamiltonian, we will first see an abstract theory on how to make an operator self-adjoint. Let us denote $\mathcal{H}$ as a Hilbert space and A as an operator acting on a domain $D(A) \subset \mathcal{H}$. In our case, Hilbert space $\mathcal{H}$ is made from the square integrable wave functions, and linear operator is given by the quantum hamiltonian H_{ws} (2.13). Adjoint operator A^* is defined by $\Psi' = A^*\Psi$ for $\Psi \in D(A^*) \subset \mathcal{H}$ satisfying

$$\langle \Psi, A\Phi \rangle = \langle \Psi', \Phi \rangle, \qquad {}^\forall\Phi \in D(A), \tag{2.25}$$

and the operator A is called symmetric if

$$\langle \Psi, A\Phi \rangle = \langle A\Psi, \Phi \rangle, \qquad {}^\forall\Phi \in D(A), \quad {}^\forall\Psi \in D(A) \subset D(A^*). \tag{2.26}$$

In particular, the operator is self-adjoint if $D(A) = D(A^*)$. In our case, we first construct the domain for H_{ws} to be symmetric, and then we extend the operator to be self-adjoint. An extension B of an operator A is defined by $D(B) \supset D(A)$ with $B = A$ on $D(A)$, and an extension B of a symmetric operator A is self-adjoint if $D(A^*) \supset D(B^*) = D(B) \supset D(A)$.

We use a generic theory concerned with self-adjoint extension of a symmetric operator (see, for example, [18]). For a symmetric operator A, we can decompose the domain $D(A^*)$ as

$$\phi = \psi + \xi_+ + \xi_-, \qquad \phi \in D(A^*), \quad \psi \in D(A), \quad \xi_+ \in K_+(A^*), \quad \xi_- \in K_-(A^*), \tag{2.27}$$

where[5]

$$K_+(A^*) := \mathrm{Ker}(A^* - i), \qquad K_-(A^*) := \mathrm{Ker}(A^* + i). \tag{2.28}$$

[4] See, for example, [18] for self-adjointness of Sturm-Liouville operators.

[5] The eigenvalues $\pm i$ could be replaced with an arbitrary pair of complex numbers c, c^* with $\mathrm{Im}\, c \neq 0$.

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Roughly speaking, if an operator A is not self-adjoint, then there are eigenfunctions of A^* whose eigenvalues have imaginary part. Then, we can see that there exists self-adjoint extensions if we can construct extensions of the symmetric operator as

$$D_U(B) = \{\phi|\phi = \psi + \xi_+ + U\xi_+, \psi \in D(A), \xi_+ \in K_+(A^*), U\xi_+ \in K_-(A^*)\} \tag{2.29}$$

with unitary transformation $U : \psi_+ \to \psi_-$. In other words, if deficiency indices (d_+, d_-) defined by

$$d_+ := \dim K_+(A^*)\,, \qquad d_- := \dim K_-(A^*) \tag{2.30}$$

are the same $d_+ = d_-$, then there exist a unitary transformation U and a family of self-adjoint extensions parametrized by U. In particular, in case of $d_+ = d_- = 0$, there is unique self-adjoint extension $B = A$.

2.5 Domain of the self-adjoint hamiltonian

Let us apply the abstract theory to our case. The Hilbert space $\mathcal{H}$ consists of the square integrable functions of $g \in \mathrm{SL}(2,\mathbb{R})$ with respect to the inner product (2.18). For the following domain of the quantum hamiltonian H_{ws} (2.13)

$$D(H_{ws}) = \{\Phi|\Phi \in \mathcal{H},\ H_{ws}\Phi \in \mathcal{H},\ U(z)|_{z=\pm 0,\pm\infty} = 0 = z\partial_z U(z)|_{z=\pm 0,\pm\infty}\}\,, \tag{2.31}$$

the hamiltonian is symmetric. This can be seen from that the symmetric condition

$$\langle H_{ws}\Phi_2, \Phi_1\rangle = \langle \Phi_2, H_{ws}\Phi_1\rangle\,, \qquad \Phi_1 \in D(H_{ws})\,, \quad \Phi_2 \in D(H_{ws}) \subset D(H^*_{ws}) \tag{2.32}$$

can be rewritten as the form of boundary conditions

$$z\left(U_2^*(z)\partial_z U_1(z) - \partial_z U_2^*(z) U_1(z)\right)\Big|_{-\infty}^{-0} - z\left(U_2^*(z)\partial_z U_1(z) - \partial_z U_2^*(z) U_1(z)\right)\Big|_{+0}^{+\infty} = 0\,. \tag{2.33}$$

Since the boundary condition (2.33) is always satisfied for ${}^\forall\Phi_1 \in D(H_{ws})$, there is no need to assign boundary conditions on $\Phi_2 \in D(H^*_{ws})$, so $D(H_{ws}) \subset D(H^*_{ws})$. As we will see below, the deficiency indices for $\lambda\mu > 0$ and $\lambda\mu < 0$ cases are different, so we investigate the self-adjointness of the symmetric hamiltonian H_{ws} (2.13) for the each case separately.

2.5.1 $\lambda\mu > 0$ case

In this case, we can see that there are no eigenfunctions $H_{ws}\Psi = \pm i\Psi$ for $\Psi \in \mathcal{H}$, so the deficiency indices are $(0,0)$. Therefore, the hamiltonian with the domain (2.31) is self-adjoint. The wave functions we have constructed (2.19) and (2.22) are included in the domain, so eigenfunctions of self-adjoint hamiltonian are given by (2.19) and (2.22) with arbitrary a_1 and b_1. The each term in the boundary condition (2.33) vanishes by itself, and there are no interactions between the patches (which is equivalent to the independence of a_1 and b_1). We denote these wave functions as $(j = 1/2 + i\omega, \omega \in \mathbb{R})$

$$\Phi^{\lambda,\mu}_\omega = \frac{\sqrt{2}(\lambda\mu)^{i\omega}}{\Gamma(2i\omega)} e^{i\lambda\gamma + i\mu\bar{\gamma}} z\, K_{2i\omega}\left(2\sqrt{\lambda\mu} e^{i\delta(z)} z\right)\,, \qquad \delta(z) = \frac{\pi}{2}(\mathrm{sgn}\ z - 1)\,. \tag{2.34}$$

We chose the normalization factors so that the two point function is given by (see, e.g., [29])

$$\langle \Phi^{\lambda_2,\mu_2}_{\omega_2}, \Phi^{\lambda_1,\mu_1}_{\omega_1} \rangle = (2\pi)^3 \delta(\lambda_1 - \lambda_2)\delta(\mu_1 - \mu_2)\left[\delta(\omega_1 - \omega_2) + R(\omega_1)\delta(\omega_1 + \omega_2)\right], \tag{2.35}$$

where

$$R(\omega) = -(\lambda\mu)^{2i\omega} \frac{\Gamma(1-2i\omega)}{\Gamma(1+2i\omega)}. \tag{2.36}$$

Physical meaning of the factor $R(\omega)$ is extracted from asymptotic behavior of the wave function near the boundary $z \to \pm 0$, equivalently, $\phi \to \infty$ (where the Liouville potential vanishes). The wave function becomes a linear combination of incident and reflected plane waves:

$$\Phi^{\lambda,\mu}_{\omega} \sim e^{i\lambda\gamma + i\mu\bar{\gamma}} e^{-\phi}\left[e^{2i\omega\phi} + R(\omega) e^{-2i\omega\phi}\right]. \tag{2.37}$$

It is evident that $R(\omega)$ denotes the reflection amplitude from the Liouville potential.

2.5.2 $\lambda\mu < 0$ case

The eigenfunctions of H_{ws} with eigenvalues $\pm i$ inside the Hilbert space $\mathcal{H}$ are given by

$$U_{k_0}(z) = \begin{cases} d_1 J_{k_0}(2\sqrt{-\lambda\mu}z) & \text{for } z > 0\,, \\ e_1 e^{-k_0\pi i} J_{k_0}(2\sqrt{-\lambda\mu}z) & \text{for } z < 0\,, \end{cases} \tag{2.38}$$

$$U_{k_0^*}(z) = \begin{cases} d_2 J_{k_0^*}(2\sqrt{-\lambda\mu}z) & \text{for } z > 0\,, \\ e_2 e^{-k_0^*\pi i} J_{k_0^*}(2\sqrt{-\lambda\mu}z) & \text{for } z < 0\,, \end{cases} \tag{2.39}$$

with $k_0 = \sqrt{1+i}$. There are two independent eigenfunctions for each eigenvalue $\pm i$, therefore the deficiency indices are $(2,2)$. Since the unitary transformation $\hat{M} : U_{k_0} \to U_{k_0^*}$ has four real parameters, there is four parameter family of self-adjoint extensions:

$$D_M(H_{ws}) = \{\Phi | \Phi = \Psi + \Xi_M, \Psi \in D(H_{ws})\}, \tag{2.40}$$

where we defined

$$\Xi_M = e^{i\lambda\gamma + i\mu\bar{\gamma}} z\, U_M(z)\,, \qquad U_M(z) = U_{k_0}(z) + \hat{M} U_{k_0}(z)\,. \tag{2.41}$$

The unitary transformation may be expressed by 2×2 unitary matrix M acting on the coefficients on (2.39) as

$$\begin{pmatrix} d_2 \\ e_2 \end{pmatrix} = M \begin{pmatrix} d_1 \\ e_1 \end{pmatrix}, \qquad M = \exp\left(i \sum_{i=0}^{3} q_i \sigma_i\right), \tag{2.42}$$

with $\sigma_0 \equiv \mathrm{diag}(1,1)$.

The self-adjointness parameters q_i fix the boundary behaviors of wave functions since the contribution from the Ξ_M part in (2.40) is dominant near the boundary. If we do not restrict the norms of the two eigenfunctions U_{k_0} and $U_{k_0^*}$ (2.39), then the domain includes rather arbitrary wave functions because the general solutions to the second order

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differential equations are given by linear combinations of two independent solutions. The symmetric condition corresponds to the boundary condition (2.33), which implies current conservation as we will see below in more detail, therefore the restriction in (2.40) is equivalent to the restriction of the wave functions to preserve the total current. There are non-trivial boundary behaviors only in the $z \to \pm\infty$ region, therefore we could say that the self-adjoint parameters determine how the current flows across the horizons.

In order to see the relation between the self-adjoint parameters and the boundary behaviors of wave functions, we set some of the parameters zero. Let us first set $q_1 = q_2 = 0$, then the wave function in the domain (2.40) behaves near the horizon as

$$U(z) \propto \frac{1}{\sqrt{z}} \cos\left(2\sqrt{-\lambda\mu z} - \frac{1}{2}\pi\left(\nu + \frac{1}{2}\right)\right), \qquad e^{i(q_0+q_3)} = -\frac{\sin\pi\,(k_0/2 - \nu/2)}{\sin\pi\,(k_0^*/2 - \nu/2)}, \tag{2.43}$$

for $z > 0$, and

$$U(z) \propto \frac{1}{\sqrt{z}} \cos\left(2\sqrt{-\lambda\mu z} - \frac{1}{2}\pi(\nu' + \frac{1}{2})\right), \qquad e^{i(q_0-q_3+2\mathrm{Im}\,k_0)} = -\frac{\sin\pi\,(k_0/2 - \nu'/2)}{\sin\pi\,(k_0^*/2 - \nu'/2)}, \tag{2.44}$$

for $z < 0$. In this case, there is no interaction between the patches with $z > 0$ and $z < 0$, and the parameters $q_0 \pm q_3$ fix the phase factors of the boundary behaviors for the each patch. In other words, there are no contributions to the each term of the boundary condition (2.33), and hence q_1, q_3 parametrize how the cancelation occurs inside the each patch $z > 0$ or $z < 0$.

For $q_i = 0\,(i \neq 1)$ the asymptotic behavior is fixed by a linear combination of

$$U(z) \propto \begin{cases} J_{k_0}(2\sqrt{-\lambda\mu z}) + e^{iq_1} J_{k_0^*}(2\sqrt{-\lambda\mu z}) & \text{for } z > 0\,, \\ e^{-k_0\pi i} J_{k_0}(2\sqrt{-\lambda\mu z}) + e^{i(q_1 - k_0^*\pi)} J_{k_0^*}(2\sqrt{-\lambda\mu z}) & \text{for } z < 0\,, \end{cases} \tag{2.45}$$

$$U(z) \propto \begin{cases} J_{k_0}(2\sqrt{-\lambda\mu z}) + e^{-iq_1} J_{k_0^*}(2\sqrt{-\lambda\mu z}) & \text{for } z > 0\,, \\ -\left(e^{-k_0\pi i} J_{k_0}(2\sqrt{-\lambda\mu z}) + e^{-i(q_1 + k_0^*\pi)} J_{k_0^*}(2\sqrt{-\lambda\mu z})\right) & \text{for } z < 0\,, \end{cases} \tag{2.46}$$

and for $q_i = 0\,(i \neq 2)$

$$U(z) \propto \begin{cases} J_{k_0}(2\sqrt{-\lambda\mu z}) + e^{iq_2} J_{k_0^*}(2\sqrt{-\lambda\mu z}) & \text{for } z > 0\,, \\ i\left(e^{-k_0\pi i} J_{k_0}(2\sqrt{-\lambda\mu z}) + e^{i(q_2 - k_0^*\pi)} J_{k_0^*}(2\sqrt{-\lambda\mu z})\right) & \text{for } z < 0\,, \end{cases} \tag{2.47}$$

$$U(z) \propto \begin{cases} J_{k_0}(2\sqrt{-\lambda\mu z}) + e^{-iq_2} J_{k_0^*}(2\sqrt{-\lambda\mu z}) & \text{for } z > 0\,, \\ -i\left(e^{-k_0\pi i} J_{k_0}(2\sqrt{-\lambda\mu z}) + e^{-i(q_2 + k_0^*\pi)} J_{k_0^*}(2\sqrt{-\lambda\mu z})\right) & \text{for } z < 0\,. \end{cases} \tag{2.48}$$

In these two cases, there are interactions between two patches, and the corresponding parameters represent how the currents flow across the horizons, or how cancelation occurs between the contributions from $z = \pm\infty$ parts in the boundary condition (2.33).

The different self-adjoint extensions correspond to different physics, and we shall adopt the self-adjoint extensions suitable to our purpose. Now that we have the eigenfunctions of the hamiltonian H_{ws} such as (2.20), (2.21), (2.23) and (2.24), we require that the domain (2.40) includes the eigenfunctions as many as possible. Later we check its physical relevance. Wave functions in the domain of self-adjoint hamiltonian must satisfy the boundary condition (2.33), and we can see which eigenfunctions are included in the domain by examining the boundary condition with $U_1(z)$ as the eigenfunctions concerned and $U_2(z)$ as (2.41). This is enough because the boundary values come from only (2.41) for wave functions in the domain (2.40) as mentioned above. However, we will examine in this way later, and first we check by using the wave functions (2.20) and (2.23) as $U_1(z)$ and $U_2(z)$ in order to see how we can obtain a maximum set of eigenfunctions in the domain. After that we will move to the case with the wave functions (2.21) and (2.24).

For the wave functions (2.20) and (2.23) we use the following notation

$$U_{2j-1}(z) = \begin{cases} J_{2j-1}(2\sqrt{-\lambda\mu} z) & \text{for } z > 0\,, \\ f(j)e^{-(2j-1)\pi i} J_{2j-1}(2\sqrt{-\lambda\mu} z) & \text{for } z < 0\,, \end{cases} \tag{2.49}$$

where $f(j) \in \mathbb{C}$ is a function of j. The total normalization can be taken arbitrary and only the relative factor is important. Utilizing the analytic continuation of the Bessel function

$$J_\nu(e^{\pi i} z) = e^{\nu\pi i} J_\nu(z)\,, \tag{2.50}$$

the boundary condition (2.33) reduces to

$$(1 + f(j_2)^* f(j_1)) \sin(\pi(j_1 - j_2)) = 0\,. \tag{2.51}$$

The above equation implies that if $j_1 - j_2 \in \mathbb{Z}$, or equivalently $j_i = m_i + j_0 + 1/2$ with $(m_i = 0, 1, 2, \ldots, 0 < j_0 \leq 1)$, then the boundary condition (2.33) is satisfied irrespective of $f(j)$. In other words, we can choose the normalization of (2.23) independent to (2.20).

In fact, the domain can include more eigenfunctions by carefully choosing $f(j)$. Here we assign the forms of $f(j)$ as

$$f(j) = \exp((\delta_c + (2j-1))\pi i)\,, \tag{2.52}$$

with $0 \leq \delta_c < 2\pi$, then we have $j_1 - j_2 \in (\mathbb{Z} + 1/2)$ as well as $j_1 - j_2 \in \mathbb{Z}$, which leads to $j_i = 1/2(m_i + \nu_0 + 1)$ $(m_i = 0, 1, 2 \ldots, 0 < \nu_0 \leq 1)$. If we use $U_1 = U_{2j-1}$ with (2.52) and $U_2 = U_M$ (2.41), then the boundary condition (2.33) reduces to the following two equations

$$\begin{aligned} (d_1^* + e^{(\delta_c+\nu_0)\pi i} e_1^*) \sin\pi\left(\frac{\nu_0}{2} - \frac{k_0^*}{2}\right) + (d_2^* + e^{(\delta_c+\nu_0)\pi i} e_2^*) \sin\pi\left(\frac{\nu_0}{2} - \frac{k_0}{2}\right) &= 0\,, \\ (d_1^* - e^{(\delta_c+\nu_0)\pi i} e_1^*) \cos\pi\left(\frac{\nu_0}{2} - \frac{k_0^*}{2}\right) + (d_2^* - e^{(\delta_c+\nu_0)\pi i} e_2^*) \cos\pi\left(\frac{\nu_0}{2} - \frac{k_0}{2}\right) &= 0\,. \end{aligned} \tag{2.53}$$

The solution to the above equations are written by using an unitary matrix as

$$\begin{pmatrix} d_2^* \\ e_2^* \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} d_1^* \\ e_1^* \end{pmatrix}, \tag{2.54}$$

$$M_{11} = M_{22} = -\frac{1}{2}\left(\frac{\sin\pi\,(\nu_0/2 - k_0^*/2)}{\sin\pi\,(\nu_0/2 - k_0/2)} + \frac{\cos\pi\,(\nu_0/2 - k_0^*/2)}{\cos\pi\,(\nu_0/2 - k_0/2)}\right), \tag{2.55}$$

$$e^{-(\delta_c+\nu_0)\pi i} M_{12} = e^{(\delta_c+\nu_0)\pi i} M_{21} = -\frac{1}{2}\left(\frac{\sin\pi\,(\nu_0/2 - k_0^*/2)}{\sin\pi\,(\nu_0/2 - k_0/2)} - \frac{\cos\pi\,(\nu_0/2 - k_0^*/2)}{\cos\pi\,(\nu_0/2 - k_0/2)}\right).$$

Thus, we can see that the self-adjoint parameters q_i $(i = 0, 1, 2, 3)$ are fixed by two parameters δ_c, ν_0. From now on, we set $\delta_c = 0$ since the constant phase factor is not relevant to the following discussions. We do not choose a particular ν_0 because it relates to the label $j = 1/2(m + \nu_0 + 1)$.

In summary, the domain we adopt has one parameter ν_0, and the domain $D_{\nu_0}(H_{ws})$ for *each* choice of ν_0 comprises wave functions taking discrete value $j = 1/2(m+\nu_0+1)$ as[6]

$$\Phi^{\lambda,\mu}_{m+\nu_0} = \sqrt{2\pi(n + \nu_0)}e^{i\lambda\gamma+i\mu\bar{\gamma}} z\, J_{m+\nu_0}\left(2\sqrt{-\lambda\mu z}\right), \tag{2.56}$$

for the both $z > 0$ and $z < 0$. The normalization is chosen so that the two point function becomes

$$\langle\Phi^{\lambda_2,\mu_2}_{m+\nu_0}, \Phi^{\lambda_1,\mu_1}_{n+\nu_0}\rangle = (2\pi)^3\delta(\lambda_1 - \lambda_2)\delta(\mu_1 - \mu_2)\delta_{m,n}\,. \tag{2.57}$$

These wave functions describe 'bound states' in the attractive Liouville potential.

Let us move to the wave functions of the type (2.21) and (2.24)

$$U_{2i\omega}(z) = \begin{cases} \alpha_\omega J_{-2i\omega}(2\sqrt{-\lambda\mu z}) + \beta_\omega J_{2i\omega}(2\sqrt{-\lambda\mu z}) & \text{for } z > 0\,, \\ \gamma_\omega J_{-2i\omega}(2\sqrt{-\lambda\mu z}) + \delta_\omega J_{2i\omega}(2\sqrt{-\lambda\mu z}) & \text{for } z < 0\,, \end{cases} \tag{2.58}$$

where we have changed the labels a_3, a_4, b_3, b_4 into $\alpha_\omega, \beta_\omega, \gamma_\omega, \sigma_\omega$ expressing $j = 1/2 + i\omega$ dependence explicitly. As before, we check the boundary condition (2.33) with $U_1 = U_{2i\omega}$ and $U_2 = U_M$ (2.41). The condition restricts the coefficients $\alpha_\omega, \beta_\omega, \gamma_\omega, \sigma_\omega$ to[7]

$$\begin{aligned} \alpha_\omega &= a_\omega e^{-\pi(\omega+i\nu_0/2)} - b_\omega e^{\pi(\omega+i\nu_0/2)}\,, \\ \beta_\omega &= b_\omega e^{-\pi(\omega-i\nu_0/2)} - a_\omega e^{\pi(\omega-i\nu_0/2)}\,, \\ \gamma_\omega &= e^{-\pi(2\omega-i\nu_0)}\left(b_\omega e^{-\pi(\omega+i\nu_0/2)} - a_\omega e^{\pi(\omega+i\nu_0/2)}\right), \\ \delta_\omega &= e^{\pi(2\omega+i\nu_0)}\left(a_\omega e^{-\pi(\omega-i\nu_0/2)} - b_\omega e^{\pi(\omega-i\nu_0/2)}\right), \end{aligned} \tag{2.59}$$

with two arbitrary functions a_ω, b_ω. This result implies that we can express $U_{2i\omega}$ by a linear combination of two independent solutions of the form (2.59). We will use the following two types of basic solutions; (1) $(a_\omega, b_\omega) = (1, 0)$ and $(0, 1)$, which are suitable to see physical meaning, and (2) $(a_\omega, b_\omega) = (1, 1)$ and $(1, -1)$, which are suitable to see mathematical meaning.

(1) $(a_\omega, b_\omega) = (1, 0)$ **and** $(0, 1)$**.** Let us first see the wave function with $(a_\omega, b_\omega) = (1, 0)$. Notice that the coefficients may be obtained by analytic continuation up to total normalization as for $z > 0$

$$K_{2i\omega}\left(e^{-\pi i/2}2\sqrt{-\lambda\mu z}\right) = \frac{\pi}{2}\frac{e^{-\pi\omega}J_{-2i\omega}(2\sqrt{-\lambda\mu z}) - e^{\pi\omega}J_{2i\omega}(2\sqrt{-\lambda\mu z})}{\sin(2i\pi\omega)}\,, \tag{2.60}$$

and for $z < 0$

$$K_{2i\omega}\left(e^{\pi i/2}2\sqrt{-\lambda\mu z}\right) = \frac{\pi}{2}\frac{e^{\pi\omega}J_{-2i\omega}(2\sqrt{-\lambda\mu z}) - e^{-\pi\omega}J_{2i\omega}(2\sqrt{-\lambda\mu z})}{\sin(2i\pi\omega)}\,. \tag{2.61}$$

[6] This result is essentially the same as that in [28].

[7] It is easier to examine the boundary condition (2.33) with $U_1 = U_{2i\omega}$ and $U_2 = U_{2j-1}$ (2.49) obtained above to get the restriction of the coefficients. The results are the same.

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For $\lambda\mu > 0$ case the wave function with (2.19) belongs to the Hilbert space since it behaves well near the horizon $z \to \infty$ as

$$\Psi_{2i\omega}\left(2\sqrt{\lambda\mu z}\right) \sim \sqrt{z}\exp\left(-iEt + iL\theta - \sqrt{L^2 - E^2}z\right), \tag{2.62}$$

where $-\lambda + \mu = E$ and $\lambda + \mu = L$ represent energy and angular momentum, respectively. That is to say, there is a damping factor $\sim e^{-\sqrt{L^2-E^2}z}$ in the wave functions. Using the analytic continuation, we have

$$\Psi_{2i\omega}\left(e^{-\pi i/2}2\sqrt{-\lambda\mu z}\right) \sim \sqrt{z}\exp\left(-iEt + iL\theta + i\sqrt{E^2 - L^2}z\right). \tag{2.63}$$

In general, the wave function may be represented as a linear combination of in-coming and out-going plane waves, but the analytic continuation lead to only an out-going plane wave $\sim e^{-iEt+i\sqrt{L^2-E^2}z}$ near the horizon $z \to \infty$. Similarly, in the $z < 0$ patch, the analytic continuation leads to only an in-coming plane wave, so the wave function with $(a_\omega, b_\omega) = (1, 0)$ shows the current flow from the the $z > 0$ patch to the $z < 0$ patch. This current flow is conserved because the wave functions satisfy the boundary condition (2.33) (even though the boundary condition originates from the self-adjointness condition of the minisuperspace hamiltonian H_{ws} (2.13)). For the case with $(a_\omega, b_\omega) = (0, 1)$, the $z < 0$ patch has an out-going plane wave and the $z > 0$ patch has an in-coming plane wave. This wave function implies the conserved current flow from the $z < 0$ patch to the $z > 0$ patch. Since the wave functions in the domain can be written as a linear combination of the two solutions, we can follow the conserved current flow through the successive patches.

In the minisuperspace model viewpoint, wave functions mean the probability of existence. Also in this context, we can show in the similar way that the wave functions in this basis imply the conserved probability flow with respect to the worldline time τ (see [28] for a discussion on the probability flow).

(2) $(a_\omega, b_\omega) = (1, 1)$ **and** $(1, -1)$**.** We first use the normalization of the wave functions as

$$\Phi^{(n)\lambda,\mu}_{\nu_0,\omega} = \sqrt{\frac{2\pi\omega}{\sinh 2\pi\omega}}e^{i\lambda\gamma + i\mu\bar{\gamma}}z\, U^{(n)}_{\nu_0,\omega}(z). \tag{2.64}$$

We denote the wave functions with $(a_\omega, b_\omega) = (1, 1)$ and $(1, -1)$ as $\Phi^{(0)\lambda,\mu}_{\nu_0,\omega}$ and $\Phi^{(1)\lambda,\mu}_{\nu_0,\omega}$, respectively. We have also defined

$$U^{(n)}_{\nu_0,\omega} = \begin{cases} J_{-2i\omega}(2\sqrt{-\lambda\mu}z) + \Theta^{(n)}_{\nu_0}(\omega)J_{2i\omega}(2\sqrt{-\lambda\mu}z) & \text{for } z > 0\,, \\ e^{\pi i(\nu_0+n)}\left(J_{-2i\omega}(2\sqrt{-\lambda\mu}e^{-\pi i}z) + \Theta^{(n)}_{\nu_0}(\omega)J_{2i\omega}(2\sqrt{-\lambda\mu}e^{-\pi i}z)\right) & \text{for } z < 0\,, \end{cases} \tag{2.65}$$

where

$$\Theta^{(n)}_{\nu_0}(\omega) = \frac{\sinh\pi(\omega - i(\nu_0 + n)/2)}{\sinh\pi(\omega + i(\nu_0 + n)/2)}. \tag{2.66}$$

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In this normalization, we can check by closely following the appendix of [26] that the wave functions in the domain $D_{\nu_0}(H_{ws})$ satisfy the completeness condition such as

$$\sum_{m=0}^{\infty}\left(\Phi^{\lambda,\mu}_{m+\nu_0}(z,\gamma,\bar\gamma)\right)^*\Phi^{\lambda,\mu}_{m+\nu_0}(z',\gamma,\bar\gamma)+$$
$$+\sum_{n=0,1}\int_0^\infty d\omega\left(\Phi^{(n)\lambda,\mu}_{\nu_0,\omega}(z,\gamma,\bar\gamma)\right)^*\Phi^{(n)\lambda,\mu}_{\nu_0,\omega}(z',\gamma,\bar\gamma) = 2\pi|z|^3\delta(z-z'), \tag{2.67}$$

as well as the orthogonality condition. This means that an arbitrary function defined in the patches with $z>0$ and $z<0$ can be decomposed by the set of the wave functions in the domain $D_{\nu_0}(H_{ws})$ for each label ν_0. If we restrict ourselves to the patch with $z>0$, then the each set of wave functions $(\Psi^{\lambda,\mu}_{2n+\nu_0},\Psi^{(0)\lambda,\mu}_{\nu_0,\omega})$ and $(\Psi^{\lambda,\mu}_{2n+1+\nu_0},\Psi^{(1)\lambda,\mu}_{\nu_0,\omega})$ with $n=0,1,2,\ldots$ and $\omega>0$ satisfies the orthogonal and complete conditions by itself. There are two sets of orthogonal and complete bases because we have to describe the functions defined in the both patches with $z>0$ and $z<0$.

In spite of this fact, we will use a different normalization as

$$\Phi^{(n)\lambda,\mu}_{\nu_0,\omega} = \frac{(-\lambda\mu)^{i\omega}}{\sqrt{2}}\Gamma(1-2i\omega)e^{i\lambda\gamma+i\mu\bar\gamma}z\,U^{(n)}_{\nu_0,\omega}(z) \tag{2.68}$$

for later convenience. Here again, we chose the normalization so that the two point function and asymptotic behavior of the wave function take the forms of (2.35) and (2.37), respectively. The corresponding reflection amplitude is readily computed:

$$R(\omega) = (-\lambda\mu)^{2i\omega}\frac{\Gamma(1-2i\omega)}{\Gamma(1+2i\omega)}\Theta^{(n)}_{\nu_0}(\omega)\,. \tag{2.69}$$

Before ending this section, let us comment on the validity of the choice of the domain $D_{\nu_0}(H_{ws})$, namely the choice of the self-adjointness parameters. First, the set of eigenfunctions in the domain gives orthogonal and complete basis as mentioned above. Second, the wave functions in the domain $D_{\nu_0}(H_{ws})$ with $\nu_0=1$ reproduces the wave functions in the single cover of AdS_3 (see, for example, [29]).[8] In particular, for $j>1/2$ states, the label j takes a half-integer value as in the single cover case. Finally, there is a probability flow between the Poincaré patches in the choice, and this is consistent with the fact that in the global coordinates the probability flows from infinite past to infinite future.

3. Primary fields and correlation functions

In the previous section, we constructed the domain $D_{\nu_0}(H_{ws})$ of the selfadjoint hamiltonian. The eigenstates in the domain correspond to the normalizable states in SL(2.$\mathbb{R}$) WZNW model. Basic quantities in WZNW model are correlation functions of primary fields, and

[8]In the single cover of AdS_3 the wave functions must be the same in the all $z<0$ patches by definition. If we require that the results with $\arg z=\pi$, which was obtained in the context, and those with $\arg z=-\pi$, which can be analyzed in a similar way, are the same, then we find the requirement reduces to $e^{2\pi i\nu_0}=1$ or $\nu_0=1$.

fields correspond to normalizable states as

$$|\lambda,\mu,j\rangle = \lim_{w\to 0}\Psi_j^{\lambda,\mu}(w,\bar{w})|0\rangle\,. \tag{3.1}$$

The ket $|\lambda,\mu,j\rangle$ at the left hand side represents a normalizable state, and the field $\Psi_j^{\lambda,\mu}$ corresponds to the state when acting to a vacuum $|0\rangle$. The worldsheet is described by the coordinates $w = e^{\tau+\sigma}$ and $\bar{w} = e^{\tau-\sigma}$, and in the minisuperspace limit, we neglect the σ-dependence.

Moreover, in the quantum mechanics, the operator corresponding to the primary field can be found from the properties under the $\mathrm{SL}(2,\mathbb{R})$ transformation as

$$\Psi_j^{\lambda,\mu}\Phi_{j'}^{\lambda',\mu'}(z,\gamma,\bar{\gamma}) := \Phi_j^{\lambda,\mu}(z,\gamma,\bar{\gamma})\Phi_{j'}^{\lambda',\mu'}(z,\gamma,\bar{\gamma})\,. \tag{3.2}$$

Therefore, the minisuperspace analogy of the multi-point correlation functions are given by the overlaps of wave functions

$$\left\langle \prod_{i=1}^{N}\Phi_{j_i}^{\lambda_i,\mu_i}(z,\gamma,\bar{\gamma})\right\rangle := \int d\gamma d\bar{\gamma}dz|z|^{-3}\prod_{i=1}^{N}\Phi_{j_i}^{\lambda_i,\mu_i}(z,\gamma,\bar{\gamma})\,. \tag{3.3}$$

In this section, we construct primary fields and compute correlation functions by making use of the $\mathrm{SL}(2,\mathbb{R})$ symmetry. In order to do so, it is easier to use a different basis from the previous one. It is given by a Fourier transform of the previous one

$$\Psi_A^j(x,\bar{x};g) = \frac{1}{2\pi^2}\int d\lambda d\mu\, e^{-i\lambda x - i\mu\bar{x}}\Phi_j^{\lambda,\mu}(g)\,. \tag{3.4}$$

This basis is convenient because the action of $\mathrm{SL}(2,\mathbb{R})$ Lie algebra are generated by differential operators, and we can see how much the $\mathrm{SL}(2,\mathbb{R})$ symmetry fixes functional form of the primary fields or correlation functions.

Since we construct the primary fields with well behaviors under the $\mathrm{SL}(2,\mathbb{R})$ action, the primary fields satisfy differential equations. In the euclidean theory ($\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ WZNW model), the solution to the differential equations is unique up to normalization (see, e.g., [10]). However, in our lorentzian theory, the solution to the differential equations is not unique, and given in a linear combination of two independent solutions. The different solution corresponds to the different wave function as in (2.21) with different a_3, a_4, and a fixed wave function corresponds to a particular linear combination of two solutions due to the state-operator correspondence. The situation is quite similar also for two point functions and three point functions. These correlation functions obey differential equations because of the $\mathrm{SL}(2,\mathbb{R})$ invariance, and in the euclidean theory, the solutions to the differential equations are almost unique, and the task is only to fix one coefficient for each [10]. However, in the lorentzian theory, the solutions to the differential equations are less restricted, and we have to compute more coefficients. In the minisuperspace limit, the correlation functions can be computed as overlaps of wave functions, so we can fix these coefficients. For the full CFT case, we may have to use the $\mathrm{SL}(2,\mathbb{R})$ symmetry to compute the correlation functions.

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As seen in the previous section, it is essential to use the both patches with the coordinate $z > 0$ and $z < 0$ for constructing the self-adjoint extension. However, in this section we only consider a patch with $z > 0$, and for this reason we will mainly use the coordinate $\phi = -\log z$. It is enough for the purpose to see how much the $\mathrm{SL}(2,\mathbb{R})$ symmetry fixes the forms of primary fields and correlation functions. Of course, it is necessary to consider also the patch with $z < 0$ in order to construct primary fields defined in the whole spacetime and to compute full correlation functions. It can be done by using the similar analysis, and the results may include some additional factors.

3.1 Primary fields and $\mathrm{SL}(2,\mathbb{R})$ symmetry

Consider again the $\mathrm{SL}(2,\mathbb{R})$ WZNW action. In the minisuperspace limit, the action is reduced to

$$S = \frac{k}{8}\int d\tau \mathrm{Tr}(g^{-1}\partial_\tau g)^2 \,. \tag{3.5}$$

In the expression, invariance under the transformation $g \to h_L g h_R$ with $h_{L,R} \in \mathrm{SL}(2,\mathbb{R})_{L,R}$ is manifest. In the parametrization (2.5), the currents associated with the symmetry are given by

$$\begin{aligned} J_L^- &= \partial_\gamma \,, & J_L^3 &= \gamma\partial_\gamma - \frac{1}{2}\partial_\phi \,, & J_L^+ &= \gamma^2\partial_\gamma - \gamma\partial_\phi - e^{-2\phi}\partial_{\bar\gamma} \,, \\ J_R^- &= \partial_{\bar\gamma} \,, & J_R^3 &= \bar\gamma\partial_{\bar\gamma} - \frac{1}{2}\partial_\phi \,, & J_R^+ &= \bar\gamma^2\partial_{\bar\gamma} - \bar\gamma\partial_\phi - e^{-2\phi}\partial_\gamma \,. \end{aligned} \tag{3.6}$$

As mentioned above, we shall use the parametrization $(x,\bar x)$ instead of (λ,μ) (in addition to j). The variables x and $\bar x$ are real-valued, so they are not related each other by complex conjugation. In the parametrization, the action of $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ Lie algebra can be generated by the following set of differential operators as

$$\begin{aligned} D_L^- &= -\partial_x \,, & D_L^3 &= -x\partial_x - (j+1) \,, & D_L^+ &= -x^2\partial_x - 2(j+1)x \,, \\ D_R^- &= -\partial_{\bar x} \,, & D_R^3 &= -\bar x\partial_{\bar x} - (j+1) \,, & D_R^+ &= -\bar x^2\partial_{\bar x} - 2(j+1)\bar x \,, \end{aligned} \tag{3.7}$$

where, as in (2.14), $c_2 = j(j-1)$ refers to the Casimir invariance of the $\mathrm{SL}(2,\mathbb{R})$ Lie algebra.

In the $\mathrm{SL}(2,\mathbb{R})$ WZNW model, primary fields are labeled by the representation of the $\mathrm{SL}(2,\mathbb{R})$ Lie algebra. In the $(x,\bar x)$ basis, the primary fields with the label j transform as

$$J^a_{L,R}\Psi^j(x,\bar x;g) = D^a_{L,R}\Psi^j(x,\bar x;g) \qquad (a = \pm, 3) \,. \tag{3.8}$$

The solution to these equations is given locally by

$$\Psi^j(x,\bar x;g) \sim [F(x,\bar x;g)]^{-2j} \qquad \text{where} \qquad F(x,\bar x;g) \equiv (\gamma - x)(\bar\gamma - \bar x)e^{\phi} + e^{-\phi} \,. \tag{3.9}$$

However, the solution has a singularity at $F(x,\bar x;g) = 0$, and needs to be prescribed suitably at such loci.

What about the euclidean AdS_3? In this case, the dynamics is described by the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZNW model, and the corresponding wave function $F(x,\bar x;g)$ is manifestly positive-definite, and hence (3.9) is the unique solution up to normalization. For the lorentzian AdS_3, however, there are singularities at $F(x,\bar{x};g)=0$, yielding

$$\Psi^j_A(x,\bar{x};g) = \begin{cases} C_j\,|F(x,\bar{x};g)|^{-2j} & \text{for} \quad F(x,\bar{x};g) > 0\,, \\ A_j C_j\,|F(x,\bar{x};g)|^{-2j} & \text{for} \quad F(x,\bar{x};g) < 0\,. \end{cases} \tag{3.10}$$

Here C_j is an overall normalization factor, while A_j is a relative phase factor.

To appreciate how A_j is determinable, consider as an example the analytic continuations, viz. $\pm i\varepsilon$ prescription. One readily finds that the primary field is given by:

$$\left[\frac{1}{F(x,\bar{x};g)\pm i\varepsilon}\right]^{2j} = \begin{cases} |F(x,\bar{x};g)|^{-2j} & \text{for } F(x,\bar{x};g) > 0\,, \\ e^{\mp 2\pi i j}|F(x,\bar{x};g)|^{-2j} & \text{for } F(x,\bar{x};g) < 0\,, \end{cases} \tag{3.11}$$

hence the phase-factor is determined as $A_j = \exp(\mp 2\pi i j)$. It can be said that the general primary field (3.10) is given by a linear combination of the basic primary fields given by the two different analytic continuations.

As another example, consider wave functions with definite parity ($(-1)^{2\epsilon}$ with $\epsilon = 0, 1/2 \bmod 1$). To extract the parity transformation rules, we find it convenient to express the function F as

$$F(x,\bar{x};g) = \begin{pmatrix} 1 & -x \end{pmatrix} g \begin{pmatrix} 1 \\ -\bar{x} \end{pmatrix}\,. \tag{3.12}$$

Then, the action of $T_h : g \to h_L g h_R$ on the primary field of definite parity is given by

$$\begin{aligned} T_h \Psi^j_{(-1)^{2\epsilon}}(x,\bar{x};g) &\equiv \Psi^j_{(-1)^{2\epsilon}}(x,\bar{x};h_L^{-1} g h_R^{-1}) \\ &= |(\beta x+\delta)(\bar{\beta}x+\bar{\delta})|^{-2j}[\mathrm{sgn}((\beta x+\delta)(\bar{\beta}x+\bar{\delta}))]^{2\epsilon} \times \\ &\quad \times \Psi^j_{(-1)^{2\epsilon}}\left(\frac{\alpha x+\gamma}{\beta x+\delta}, \frac{\bar{\alpha}\bar{x}+\bar{\gamma}}{\bar{\beta}\bar{x}+\bar{\delta}}; g\right)\,, \end{aligned} \tag{3.13}$$

where

$$h_L = \begin{pmatrix} \alpha & \lambda \\ \beta & \delta \end{pmatrix}\,, \qquad h_R = \begin{pmatrix} \bar{\alpha} & \bar{\lambda} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}\,. \tag{3.14}$$

Therefore, from the parity transformation $g \to -g$, we can see that the wave function with parity $+1$ is given by

$$\Psi^j_{+1}(x,\bar{x};g) \sim |F(x,\bar{x};g)|^{-2j}\,, \tag{3.15}$$

or (3.10) with $A_j = +1$ and the one with parity -1 is given by

$$\Psi^j_{-1}(x,\bar{x};g) \sim \mathrm{sgn}(F(x,\bar{x};g))|F(x,\bar{x};g)|^{-2j}\,. \tag{3.16}$$

or (3.10) with $A_j = -1$. The general primary field with A_j can be written as a linear combination of the primary fields with parity ± 1, and this fact will be found useful to compute the three point functions.

3.2 Relation to the normalizable states

Because of the state-operator correspondence, we can compare the primary fields defined above with the eigenfunctions included in the domain $D_{\nu_0}(H_{ws})$. To see the relation, we have to take a care about the difference of the bases. In fact, the primary fields in the $(x,\bar{x})$ basis (3.10) are Fourier transforms of the wave functions in the (λ,μ) basis[9]

$$\Psi_j^{\lambda,\mu}(g) = \int dx d\bar{x}\, e^{i\lambda x + i\mu\bar{x}}\, \Psi_A^j(x,\bar{x};g)\,. \tag{3.17}$$

Demanding them to match with the wave functions obtained in section 2, we will be able to extract the 'phase-factor' A_j. The inverse of the above Fourier transform (3.4) is given by integrating over the whole ranges of λ and μ, so A_j should not depend on the sign of $\lambda\mu$.

We should remark that the above integral converges only if $1/2 < \mathrm{Re}\, j < 3/4$. For $j = 1/2 + i\omega$ case, we should first introduce a regulator $j = 1/2 + \epsilon + i\omega$, and then take the limit of $\epsilon \to +0$. For $j > 1/2$ case, we can only compare the two representations within a small range $1/2 < j < 3/4$ to fix A_j, and for $j \geq 3/4$ we will use the analytic continuation on j as A_j.

Let us first consider $\lambda\mu > 0$ case. This is the case already present in the euclidean theory. We find that the integral (3.17) becomes

$$\Psi_j^{\lambda,\mu}(g) = C_j \left[\frac{\cos\pi(1-2j) + A_j}{(2j-1)\sin\pi(1-2j)}\right] \frac{2\pi(\lambda\mu)^{j-1/2}}{\Gamma(2j-1)} e^{i\lambda\gamma + i\mu\bar{\gamma}} e^{-\phi} K_{2i\omega}\left(2\sqrt{\lambda\mu}e^{-\phi}\right)\,. \tag{3.18}$$

Hence, we recover precisely the result (2.34) *provided* we choose the normalization coefficient $(j = 1/2 + i\omega)$

$$C_j \equiv \frac{1}{\sqrt{2}\pi}\left[\frac{\omega\sinh(2\omega\pi)}{\cosh(2\omega\pi) + A_j}\right]\,. \tag{3.19}$$

Here we should notice that there appears A_j dependence only in the total normalization.

On the other hand, in $\lambda\mu < 0$ case, there is no euclidean counterpart. The Fourier transform (3.17) in the present case is

$$\begin{aligned}\Phi_j^{\lambda,\mu}(g) = C_j \frac{\pi^2(-\lambda\mu)^{j-1/2}}{\Gamma(2j)\sin^2\pi(1-2j)} e^{i\lambda\gamma + i\mu\bar{\gamma}} e^{-\phi} \Big(& (\cos\pi(1-2j) + A_j) J_{1-2j}\left(2\sqrt{-\lambda\mu}e^{-\phi}\right) - \\ & - (\cos\pi(1-2j)\cdot A_j + 1) \times \\ & \times J_{2j-1}\left(2\sqrt{-\lambda\mu}e^{-\phi}\right)\Big)\,.\end{aligned} \tag{3.20}$$

When $j = 1/2 + i\omega$, the normalization is set by (3.19) and we have

$$\begin{aligned}\Phi_\omega^{\lambda,\mu}(g) &= \frac{(-\lambda\mu)^{i\omega}}{\sqrt{2}}\Gamma(1-2i\omega) e^{i\lambda\gamma + i\mu\bar{\gamma}} e^{-\phi} \times \\ &\times \left(J_{-2i\omega}\left(2\sqrt{-\lambda\mu}e^{-\phi}\right) - \frac{\cosh(2\omega\pi)A_j + 1}{\cosh(2\omega\pi) + A_j} J_{2i\omega}\left(2\sqrt{-\lambda\mu}e^{-\phi}\right)\right)\,.\end{aligned} \tag{3.21}$$

[9]We define the integration measure as $dx d\bar{x} \equiv dydz$ with $x = y + z, \bar{x} = y - z$.

Requiring that the result be (2.68), we find the following relation between $\Theta^{(n)}_{\nu_0}$ and A_j:

$$\Theta^{(n)}_{\nu_0}(\omega) = -\frac{\cosh(2\omega\pi)A_j + 1}{\cosh(2\omega\pi) + A_j}, \quad \text{equivalently}\,, \quad A_j = -\frac{\cosh\pi(\omega - i(\nu_0 + n)/2)}{\cosh\pi(\omega + i(\nu_0 + n)/2)}\,. \tag{3.22}$$

As a checkpoint, take again the two examples considered in the previous section. For these examples, the forms of Θ ($= \Theta^{(n)}_{\nu_0}$ in the above example) and A_j are known already. For the example of analytic continuations, $\Theta = -\exp(\mp 2\omega\pi)$ (see (2.60)), and (3.22) correctly reproduces $A_j = -\exp(\mp 2\omega\pi)$ in (3.11). For the example of parity eigenstates, $A_j = +1$ and $A_j = -1$. It then follows from (3.22) that $\Theta = -1$ and $\Theta = +1$, respectively, and these are the wave functions with definite parity as in [32, 29].

For $1/2 < j < 3/4$ case, if we use in (3.20)

$$C_j = \frac{\sqrt{2\pi(2j-1)}\Gamma(2j)}{\pi^2(-\lambda\mu)^{j-1/2}}\,, \qquad A_j = -\cos\pi(1-2j)\,, \tag{3.23}$$

then we reproduce the wave function (2.56). We can show as a consistency check that the integral (3.17) for $\lambda\mu > 0$ vanishes if we use (3.23). For $j \geq 3/4$ case, we cannot perform the Fourier transform (3.17) because the integral diverges. Therefore, we *define* the primary fields with $j \geq 3/4$ corresponding to the states by using the parameters (3.23) with $j \geq 3/4$.

3.3 Two point functions

We mainly consider $j = 1/2 + i\omega$ case. For the primary fields (3.10) with the coefficients (3.19) and (3.22), asymptotic behavior near the boundary $z \to 0$, equivalently, $\phi \to \infty$ is extracted:

$$\Psi^j_A(x,\bar{x};g) \sim \delta(\gamma - x)\delta(\bar{\gamma} - \bar{x})e^{(2j-2)\phi} + C_j|(\gamma - x)(\bar{\gamma} - \bar{x})|^{-2j}_A e^{-2j\phi}\,. \tag{3.24}$$

Here, we abbreviated

$$|(\gamma - x)(\bar{\gamma} - \bar{x})|^{-2j}_A = \begin{cases} |(\gamma - x)(\bar{\gamma} - \bar{x})|^{-2j} & \text{for } (\gamma - x)(\bar{\gamma} - \bar{x}) > 0\,, \\ A_j|(\gamma - x)(\bar{\gamma} - \bar{x})|^{-2j} & \text{for } (\gamma - x)(\bar{\gamma} - \bar{x}) < 0\,. \end{cases} \tag{3.25}$$

The second term in (3.24) represents the reflection amplitude, and we have the reflection relation:

$$\Psi^j_A(x,\bar{x};g) = C_j \int dx' d\bar{x}' |(x - x')(\bar{x} - \bar{x}')|^{-2j}_A \Psi^{1-j}_A(x', \bar{x}'; g)\,. \tag{3.26}$$

This relation is verifiable by utilizing the Mellin transformations. Because of this relation, we can restrict the spectrum to $\omega > 0$, and this truncation is consistent with the completeness condition (2.67).

The wave functions with $j > 1/2$ represent the bound states, and hence there is no such a reflection relation. This is consistent with the fact that wave functions with $1 - j$ are not included in the Hilbert space.

Two point function can be extracted from the reflection relation (3.26) or from the Fourier transform of the result in the previous section. Functional form of the two point function obtained so is restricted by the $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ symmetry. Since the two point functions ought to be invariant under the transformations generated by the differential operators (3.7), we obtain the following ansatz ($x_{12} \equiv x_1 - x_2$):

$$\begin{aligned}\left\langle \Psi_A^{j_2}(x_2,\bar{x}_2)\Psi_A^{j_1}(x_1,\bar{x}_1)\right\rangle &= N_{j_1}\delta(j_1+j_2-1)\delta(x_{12})\delta(\bar{x}_{12}) + \delta(j_1-j_2)\times \\ &\times \sum_{\eta,\bar{\eta}=0,1/2} D_{j_1}^{\eta\bar{\eta}}(\operatorname{sgn} x_{12})^{2\eta}(\operatorname{sgn}\bar{x}_{12})^{2\bar{\eta}}|x_{12}|^{-2j_1}|\bar{x}_{12}|^{-2j_1}\,. \end{aligned} \tag{3.27}$$

Notice that four possible solutions are emerging, as shown in the second term, though there is only one solution possible in the euclidean theory.

Since the Fourier transform of the two point function is already obtained in the previous subsection, the coefficients N_j and $D_j^{\eta,\bar{\eta}}$ can be determined as

$$\left\langle \Psi_A^{\omega_2}(x_2,\bar{x}_2)\Psi_A^{\omega_1}(x_1,\bar{x}_1)\right\rangle = 4\pi\delta(x_{12})\delta(\bar{x}_{12})\delta(\omega_1+\omega_2)+4\pi C_{\omega_1}|x_{12}\bar{x}_{12}|_A^{-2j}\delta(\omega_1-\omega_2)\,. \tag{3.28}$$

It is quite significant that we find the *identical* result (including normalization factors) for both $\lambda\mu > 0$ and $\lambda\mu < 0$ branches. Fourier transform of the second term in (3.28) yields precisely the reflection function $R(\omega)$ (2.36) and (2.69) for $\lambda\mu > 0$ and $\lambda\mu < 0$, respectively.

For $j > 1/2$ case, the two point function is simply given by

$$\left\langle \Psi^{n_2+\nu_0}(x_2,\bar{x}_2)\Psi^{n_1+\nu_0}(x_1,\bar{x}_1)\right\rangle = 4\pi\delta(x_{12})\delta(\bar{x}_{12})\delta_{n_1,n_2}\,. \tag{3.29}$$

There is only one term involved, which is related to the fact that there is no reflection relation in this case.

3.4 Three point functions

In this subsection, we compute three point functions. As in the primary operators and the two point functions, the $\mathrm{SL}(2,\mathbb{R})$ symmetry restricts functional form of three point functions. However, if we only solve the differential equations coming from the requirement of the $\mathrm{SL}(2,\mathbb{R})$ invariance, then the solution to the equation has too many undetermined coefficients. Therefore, we also utilize the properties under the parity transformation. In order to do this, we first decompose the primary field (3.10) by the fields with definite parity $A = \pm 1$

$$\Psi_A^j(x,\bar{x};g) = P_A^0\Psi_{+1}^j(x,\bar{x};g) + P_A^{1/2}\Psi_{-1}^j(x,\bar{x};g)\,. \tag{3.30}$$

Then, the general three point function can be obtained as

$$\left\langle \prod_{i=1}^3 \Psi_{A_i}^{j_i}(x_i,\bar{x}_i)\right\rangle = \sum_{\substack{\epsilon_1,\epsilon_2=0,1/2 \\ \epsilon_3=\epsilon_1+\epsilon_2 \bmod 1}} P_{A_1}^{\epsilon_1}P_{A_2}^{\epsilon_2}P_{A_3}^{\epsilon_3}\left\langle \prod_{i=1}^3 \Psi_{(-1)^{2\epsilon_i}}^{j_i}(x_i,\bar{x}_i)\right\rangle\,. \tag{3.31}$$

Here we have used the parity conservation $\epsilon_1+\epsilon_2+\epsilon_3 = 0 \bmod 1$.

The primary fields with definite parity transform under the $\mathrm{SL}(2,\mathbb{R})$ action as in (3.13). Therefore, by considering the three point functions of primary fields of this type, we can deduce functional form of the three point functions via these symmetries as in [32, 29]:

$$\left\langle \prod_{i=1}^{3} \Psi^{j_i}_{(-1)^{2\epsilon_i}}(x_i,\bar{x}_i) \right\rangle = \sum_{\omega,\bar{\omega}=0,1/2} C_{\omega,\bar{\omega}}(j_i) K_\omega(x_i) \bar{K}_{\bar{\omega}}(\bar{x}_i)\,, \tag{3.32}$$

where

$$\begin{aligned} K_\omega(x_i) &= (\mathrm{sgn} x_{12})^{2\eta_1}|x_{12}|^{-\tilde{j}_1}(\mathrm{sgn} x_{23})^{2\eta_2}|x_{23}|^{-\tilde{j}_2}(\mathrm{sgn} x_{31})^{2\eta_3}|x_{31}|^{-\tilde{j}_3}\,, \\ \bar{K}_{\bar{\omega}}(\bar{x}_i) &= (\mathrm{sgn}\bar{x}_{12})^{2\bar{\eta}_3}|\bar{x}_{23}|^{-\tilde{j}_1}(\mathrm{sgn}\bar{x}_{23})^{2\bar{\eta}_2}|\bar{x}_{23}|^{-\tilde{j}_2}(\mathrm{sgn}\bar{x}_{31})^{2\bar{\eta}_3}|\bar{x}_{31}|^{-\tilde{j}_3}\,, \end{aligned} \tag{3.33}$$

$$\eta_i = \epsilon_i + \omega \bmod 1\,, \quad \bar{\eta}_i = \epsilon_i + \bar{\omega} \bmod 1\,, \quad \tilde{j} = j_1 + j_2 + j_3\,, \quad \tilde{j}_i = \tilde{j} - 2j_i \tag{3.34}$$

In the euclidean theory, the three point function is determined up to j_i-dependent normalization. However, in the lorentzian case, the normalization may be changed when x_{ij} crosses the zero. Nevertheless, for the three point functions under consideration, we were able to further restrict their functional form into only four independent ones with the label $\omega, \bar{\omega} = 0, 1/2$.

In the following, we determine the coefficients $C_{\omega,\bar{\omega}}$ which cannot be determined only from the group theoretic consideration. For the purpose, we again compute the Fourier transform of the above quantities, and compare with overlaps of the three wave functions given in section 2. In the latter basis, we can perform the integral more easily because it is merely the integral of three Bessel functions, however we cannot see how much the $\mathrm{SL}(2,\mathbb{R})$ symmetry determines the functional form of three point function.

Here we should remark that we can perform the Fourier transform only if $\mathrm{Re}\,\tilde{j} < 2, \mathrm{Re}\,\tilde{j}_i > 0$ $(i = 1,2,3)$. For $j_i = 1/2 + i\omega_i$ $(i = 1,2,3)$, this condition is always satisfied, but if $j_i > 1/2$ for some $i = 1,2,3$, then the condition may be violated. If the condition is violated, then we use the analytic continuation on j_i as the coefficients $C_{\omega,\bar{\omega}}(j_i)$. For simplicity, we only consider the case with $j = 1/2 + i\omega$. The case with $j > 1/2$ can be obtained by setting $\omega = -i(j - 1/2)$ and using the different normalization (3.23). First, we compare in the case with $\lambda_i\mu_i > 0$ $(i = 1,2,3)$ to fix the coefficients $C_{\omega,\bar{\omega}}(j_i)$. The Fourier transform of the primary field in this case does not depend on the choice of A_j, and we can fix the coefficients completely. Then, we compare in the case with $\lambda_i\mu_i < 0$ $(i = 1,2,3)$ as a consistency check.

3.4.1 $\lambda_i\mu_i > 0$ $(i = 1,2,3)$ case

Let us first see the overlaps of the three wave functions of the type (2.34). Overlap integral of three Bessel functions is computable by utilizing known results, for example, the appendix A of [29]. Making use of integral representations of the hypergeometric functions

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collected in appendix A.2, we found the three point function is given by[10]

$$\left\langle \prod_{i=1}^{3} \Phi_{\omega_i}^{\lambda_i,\mu_i} \right\rangle = \frac{\pi}{\sqrt{2}} \delta(\lambda_1+\lambda_2+\lambda_3)\delta(\mu_1+\mu_2+\mu_3)(\lambda_3\mu_3)^{i\tilde{\omega}-1/2}\Gamma\left(\frac{1}{2}+i\tilde{\omega}\right) \times$$
$$\times \prod_{j=1}^{3} \frac{\Gamma(1/2-i\tilde{\omega}_j)}{\Gamma(2i\omega_j)} \left(\frac{\sin\pi(1/2+i\tilde{\omega})\sin\pi(1/2+i\tilde{\omega}_2)}{\sin\pi(2i\omega_2)} I_1\bar{I}_1 - \right.$$
$$\left. - \frac{\sin\pi(1/2+i\tilde{\omega}_1)\sin\pi(1/2+i\tilde{\omega}_3)}{\sin\pi(2i\omega_2)} I_2\bar{I}_2 \right) . \quad (3.35)$$

Here, we abbreviated combinations of ω-quantum numbers as

$$\tilde{\omega} = \omega_1 + \omega_2 + \omega_3 \qquad \text{and} \qquad \tilde{\omega}_j = \tilde{\omega} - 2\omega_j . \quad (3.36)$$

We also shorthanded the integrals

$$I_{(1,2)} = I_{(1,2)}\left(-\frac{1}{2}+i\tilde{\omega}_1, -\frac{1}{2}+i\tilde{\omega}_2; -\frac{1}{2}+i\tilde{\omega}_3; -\frac{\lambda_2}{\lambda_3}\right), \quad (3.37)$$

and the barred ones $\bar{I}_{(1,2)}$ by replacing $-\lambda_2/\lambda_3$ with $-\mu_2/\mu_3$.

We already know that the Fourier transform of the wave function with $A_j = \pm 1$ is given by (2.34) independent of the parity. Therefore, we can determine the coefficients $C_{\omega,\bar{\omega}}(j_i)$ by comparing the Fourier transform of (3.32) with (3.35). We find that

$$\left\langle \prod_{i=1}^{3} \Psi^{j_i}_{(-1)^{2\epsilon_i}}(x_i,\bar{x}_i) \right\rangle = \frac{\pi}{16\sqrt{2}} \Gamma(\tilde{j}-1) \prod_{i=1}^{3} \frac{1}{\sin\pi(j_i+\epsilon_i)\Gamma(2j_i-1)\Gamma(1-\tilde{j}_i)} \times$$
$$\times \left(\sum_{\omega=0,1/2} \frac{(-1)^{2\omega}\sin\pi(\frac{1}{2}\tilde{j}+\omega)}{\prod_{i=1}^{3}\sin\pi(\frac{1}{2}\tilde{j}_i+\eta_i)} K_\omega(x_i)\bar{K}_\omega(\bar{x}_i) \right) . \quad (3.38)$$

We should remark that the coefficients $C_{\omega,\bar{\omega}}(j_i)$ depend on the parity of the wave functions. Therefore, for the general three point functions (3.31), the coefficients $C_{\omega,\bar{\omega}}(j_i)$ depend on the parameters A_j of the primary fields (3.10).

3.4.2 $\lambda_i\mu_i < 0$ $(i = 1, 2, 3)$ case

The overlaps of the three wave functions of the type (2.68) was essentially obtained in [29]. There, the wave functions of the type (2.68) with only the phases $\Theta^{(0)}_{\nu_0=1} = -1$ and $\Theta^{(1)}_{\nu_0=1} = +1$ were considered. These wave functions are the ones with the definite parity with ± 1. Denote these two types of wave function as $\Phi^{\lambda,\mu}_{0,\omega}$ and $\Phi^{\lambda,\mu}_{1/2,\omega}$, respectively, where the label $\epsilon(= 0, 1/2)$ keeps track of the parity $(-1)^{2\epsilon}$. As before, the general wave functions (2.68) are expressible as a linear combination of the two wave functions of definite parity as

$$\Phi^{(n)\lambda,\mu}_{\nu_0,\omega} = P^{(n)0}_{\nu_0}\Phi^{\lambda,\mu}_{0,\omega} + P^{(n)1/2}_{\nu_0}\Phi^{\lambda,\mu}_{1/2,\omega} . \quad (3.39)$$

[10] We set $\lambda_1\mu_2 \geq \lambda_2\mu_1$ without loss of generality.

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It follows immediately that the general three point functions are expressed as

$$\left\langle \prod_{i=1}^{3} \Phi^{(n_i)\lambda_i,\mu_i}_{\nu_{0,i},\omega_i} \right\rangle = \sum_{\substack{\epsilon_1,\epsilon_2=0,1/2 \\ \epsilon_3=\epsilon_1+\epsilon_2 \bmod 1}} P^{(n_1)\epsilon_1}_{\nu_{0,1}} P^{(n_2)\epsilon_2}_{\nu_{0,2}} P^{(n_3)\epsilon_3}_{\nu_{0,3}} \left\langle \prod_{i=1}^{3} \Phi^{\lambda_i,\mu_i}_{\epsilon_i,\omega_i} \right\rangle . \tag{3.40}$$

Define the following integrals

$$C_\omega(\lambda_i) = \int dx_1 dx_2 dx_3 K_\omega(x_i) e^{i\lambda_1 x_1 + i\lambda_2 x_2 + i\lambda_3 x_3} , \tag{3.41}$$

then the three point functions (3.40) are written in terms of these integrals as [29][11]

$$\left\langle \prod_{i=1}^{3} \Phi^{\lambda_i,\mu_i}_{\epsilon_i,\omega_i} \right\rangle = \frac{(-1)^{2\epsilon_3}}{64\sqrt{2}} \prod_{i=1}^{3} \frac{|\lambda_i|^{2i\omega_i} e^{\pi i \epsilon_i}}{\sin \pi(1/2+\omega_i+\epsilon_i)\Gamma(2i\omega_i)} \left(\sum_{\omega=0,1/2} (C_\omega(\lambda_i))^* \, C_\omega(\mu_i) \right) . \tag{3.42}$$

One can check that these Fourier transforms reproduce (3.38). In this expression, the latter factor may be determined purely from group theoretic analysis [29], and the former factor comes from the normalization of wave functions (2.64).

4. Conclusion

In this paper, we investigated string dynamics on lorentzian AdS_3 in the minisuperspace limit. We constructed Hilbert space by the normalizable wave functions, and found that the hamiltonian is given by a differential operator. In order to see the self-adjointness of the hamiltonian, we have to also determine the domain on which the hamiltonian acts. For $\lambda\mu > 0$ case there is unique self-adjoint hamiltonian, however for $\lambda\mu < 0$ case there is four parameter family of self-adjoint extensions, and we picked up a domain labeled by one parameter ν_0. The condition of the self-adjointness reduces to the boundary condition of the wave functions as (2.33), which can be interpreted as the condition of the probability conservation or the current conservation.

We have constructed the Hilbert space by the square integrable functions with respect to the inner product (2.18), and because of the inner product, the eigenfunctions of the type (2.20) have only $j > 1/2$. It is known [33] that there are two types of solutions to the Klein-Gordon equation (2.14) for $0 < j < 1$ with the same Casimir invariance $c_2 = j(j-1)$; one is given by J_{2j-1} as in (2.20) and the other is J_{1-2j}. In order to include the both of the solutions to the Hilbert space, we have to modify the inner product as suggested in [13] (but we did not because the unmodified inner product (2.18) gives the two point function of the minisuperspace limit of the WZNW model). The ambiguity is related to the fact that the AdS space is non-globally hyperbolic,[12] and one parameter

[11] Without loss of generality, we set $\lambda_1, \lambda_2 > 0$ and $\lambda_3 < 0$.

[12] In the Hilbert space we used, wave functions vanish sufficiently fast near the boundary $z \to \pm 0$, where the non-globally hyperbolic property of AdS space is important. Therefore, the every ambiguities dealed with in this paper are related to the existence of the horizons in the Poincaré patches of lorentzian AdS_3, and not related to the non-globally hyperbolic property.

family of self-adjoint extension is constructed in both the Poincaré patch [34] and the global patch [35]. The application of the ambiguity to the AdS/CFT correspondence is discussed in [36].

Although we mainly considered in the Poincaré coordinates, the similar analysis can be done in the global coordinates. As mentioned in [30], the solution to the Klein-Gordon equation is unique if we require it behaves well near the center and the boundary. In fact, by closely following the analysis in $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZNW case [10], we can show that there is only unique self-adjoint extension of the hamiltonian. The difference from the euclidean case is that there appears discrete spectrum in $j > 1/2$, in addition to continuous spectrum in $j = 1/2 + i\mathbb{R}$, which also exists in the euclidean case.

The choice of coordinate system must not change the physics, so the wave functions defined in the whole Poincaré patches should be equivalent to the wave functions in the global coordinates. In other words, we should determine how to connect wave functions defined in each Poincaré patch at the horizon so that the wave functions reproduce the ones defined in the whole spacetime described by the global coordinates. In this context, we could say that we did it by assigning the self-adjointness of the hamiltonian. For instance, now we have the discrete label $j = 1/2(n + \nu_0 + 1)$ with $n = 0, 1, \cdots$ for the wave functions with $j > 1/2$ (2.56). In the global coordinates, the conserved charges are related to the eigenvalues $(m, \bar{m})$ of J^3_R and J^3_L. For the states with $j > 1/2$ the quantum number j is related to m as $j = m + n$ or $j = -m + n$ with $n \in \mathbb{Z}$. For the single cover of AdS_3, the label m takes $m \in \frac{1}{2}\mathbb{Z}$ due to the closed timelike curve, and hence j is also half integer. For the universal cover, we unwrap the closed timelike curve, and the label m is replaced by $m = 1/2(n + \nu)$ with $0 < \nu \leq 1$, where ν is integrated out later. Therefore, the label j takes the same discrete value $j = 1/2(n + \nu_0 + 1)$.[13]

In the Poincaré patch, there is no particular vacuum, and we have to take care of the connection of the wave functions at the horizon. The situation may be similar to the black hole case. It is interesting to see if we can connect the wave functions between horizons even in the case of black hole. Since the BTZ black hole can be obtained by orbifolding the AdS_3 spacetime, we may be able to directly apply our analysis. If the black hole background is asymptotically AdS, then we can apply our analysis to the AdS/CFT correspondence. Recently, in [37]–[41], it was proposed that the information inside the horizon is obtainable from the boundary CFT viewpoint. They use the analytic continuation, and our analysis may be useful if we want to investigate in more general vacua.

Among the four parameter family of the self-adjoint extension, we pick up a one parameter family in order to obtain the domain suitable to our case. Apart from the choice used in the context, we may use a sensible choice made in [27] and [26].[14] Features of the choice are that there are no interaction between the patches and that the wave functions given by the analytic continuation from $\lambda\mu > 0$ to $\lambda\mu < 0$ are not included in the domain of self-adjoint hamiltonian. In [26], the timelike Liouville theory is defined using only a patch. Thus, if we prepare two patches and glue at the point where the expectation value

[13] I am grateful to Y. Satoh for pointing out this fact.

[14] The self-adjoint parameters are set $q_1 = q_2 = q_3 = 0$ in this case.

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of the tachyon diverges, then the analytic continuation from the spacelike Liouville theory may work as in [19, 20] and [23]–[25]. It would be interesting to pursue this issue in more detail.

Moreover, we computed two and three point functions involving primary fields. We used $(x, \bar{x})$ representation since the $\mathrm{SL}(2,\mathbb{R})$ actions are expressible as differential operators (3.7). Using the property under the $\mathrm{SL}(2,\mathbb{R})$ transformation, the primary fields can be given as solutions to the differential equations (3.8). The general solutions are given by (3.10), which has one parameter A_j in addition to the overall normalization C_j. As we can see in (3.20), there are maps between A_j, C_j and $a_{3,4}$ in (2.21). In the euclidean theory, the solution to (3.8) is unique up to normalization C_j and the appearance of another parameter A_j is a new feature in the lorentzian case. Functional forms of two and three point functions are also fixed by the $\mathrm{SL}(2,\mathbb{R})$ symmetry as (3.27) and (3.32). These solutions have several undetermined coefficients contrast to the fact that there is only one undetermined coefficient in the euclidean case. Since the correlation functions in the minisuperspace model are given by overlaps of the wave functions, we can compute them in the minisuperspace approximation as (3.28) and (3.38). Correlation functions in the full CFT may be computable if we use the bootstrap constraint as in the euclidean case [11] as well as the forms of the solutions obtained in this paper.

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A. Several useful formulae

A.1 Integrals and related formulae

Useful integrals reducing to the Gamma function are

$$\int_0^a dx (a^2 - x^2)^b = a^{1+2b}\sqrt{\pi}\frac{\Gamma(1+b)}{2\Gamma(3/2+b)} \qquad [a > 0, \mathrm{Re}\, b > -1]\,, \tag{A.1}$$

$$\int_a^\infty dx (x^2 - a^2)^b = a^{1+2b}\frac{\Gamma(-1/2-b)\Gamma(1+b)}{2\sqrt{\pi}} \qquad \left[a > 0, -1 < \mathrm{Re}\, b < -\frac{1}{2}\right], \tag{A.2}$$

$$\int_0^\infty dx (a^2 + x^2)^b = a^{1+2b}\sqrt{\pi}\frac{\Gamma(-b-1/2)}{2\Gamma(-b)} \qquad \left[\mathrm{Re}\, b < -\frac{1}{2}\right], \tag{A.3}$$

$$\int_0^\infty dx \frac{x^{a-1}}{(x+1)^{a+b}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \qquad [a, b > 0]\,, \tag{A.4}$$

$$\int_1^\infty dx x^{a-1}(x-1)^{b-1} = \frac{\Gamma(1-a-b)\Gamma(b)}{\Gamma(1-a)} \qquad [a+b < 1, b > 0]\,. \tag{A.5}$$

We use the following relations of the Gamma matrix as

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}\,, \qquad \Gamma(2z) = \frac{2^{2z}}{2\sqrt{\pi}}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right). \tag{A.6}$$

Fourier transforms used in the context are

$$
\begin{aligned}
\int_{-\infty}^{\infty} dx \frac{1}{|x|^\nu} e^{-ixy} &= 2\sin\left(\frac{\nu\pi}{2}\right) \frac{\Gamma(1-\nu)}{|y|^{1-\nu}} \qquad [0 < \mathrm{Re}\,\nu < 1]\,, &\text{(A.7)}\\
\int_{-\infty}^{\infty} dx \frac{\mathrm{sgn}x}{|x|^\nu} e^{-ixy} &= 2i\mathrm{sgn}y \cos\left(\frac{\nu\pi}{2}\right) \frac{\Gamma(1-\nu)}{|y|^{1-\nu}} \qquad [0 < \mathrm{Re}\,\nu < 2, \nu \neq 1]\,, &\text{(A.8)}\\
\int_{-\infty}^{\infty} dx \frac{1}{(x^2+a^2)^{\nu+\frac{1}{2}}} e^{-ixy} &= \frac{2\sqrt{\pi}}{\Gamma(\nu+1/2)} \left|\frac{y}{2a}\right|^\nu K_\nu(a|y|) \qquad \left[\mathrm{Re}\,\nu > -\frac{1}{2}\right], &\text{(A.9)}\\
\int_{|x|<a} dx \frac{1}{(a^2-x^2)^{\nu+\frac{1}{2}}} e^{-ixy} &= \sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right) \left|\frac{y}{2a}\right|^\nu J_{-\nu}(a|y|) \qquad \left[\mathrm{Re}\,\nu < \frac{1}{2}\right], &\text{(A.10)}\\
\int_{|x|>a} dx \frac{1}{(x^2-a^2)^{\nu+\frac{1}{2}}} e^{-ixy} &= -\sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right) \left|\frac{y}{2a}\right|^\nu N_\nu(a|y|) \qquad \left[-\frac{1}{2} < \mathrm{Re}\,\nu < \frac{1}{2}\right]\\
N_\nu(z) &= \frac{1}{\sin\nu\pi}[\cos\nu\pi J_\nu(z) - J_{-\nu}(z)]\,. &\text{(A.11)}
\end{aligned}
$$

The asymptotic forms of Bessel functions are

$$
J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{1}{2}\pi(\nu+\frac{1}{2})\right) \ (z\sim\infty)\,, \qquad J_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu (z\sim 0)\,. \tag{A.12}
$$

The asymptotic expansion of modified Bessel function for large $|z|$ is

$$
K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n+1/2)}{n!\ \Gamma(\nu-n+1/2)(2z)^{2n}}\,. \tag{A.13}
$$

A.2 Hypergeometric functions

We use the following integrals as $(0 < z < 1)$

$$
\begin{aligned}
I_1(a,b,c;z) &= \int_1^\infty dw w^a (w-1)^b (w-z)^c \quad [\mathrm{Re}\ (-a-c) > \mathrm{Re}\ (-a-b-c-1) > 0]\\
&= \frac{\Gamma(-a-b-c-1)\Gamma(b+1)}{\Gamma(-a-c)} F(-c, -a-b-c-1; -a-c; z)\,, &\text{(A.14)}\\
I_2(a,b,c;z) &= \int_0^z dw w^a (1-w)^b (z-w)^c \quad [\mathrm{Re}\ (a+c+2) > \mathrm{Re}\ (a+1) > 0]\\
&= z^{1+a+c} \frac{\Gamma(a+1)\Gamma(c+1)}{\Gamma(a+c+2)} F(-b, a+1; a+c+2; z)\,, &\text{(A.15)}
\end{aligned}
$$

where $F(a,b;c;z)$ is the hypergeometric function. Using a formula for the hypergeometric function

$$
\begin{aligned}
\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a,b;c;z) &= \Gamma(a+b-c)(1-z)^{c-a-b} F(c-a, c-b; c-a-b+1; 1-z) +\\
&\quad + \frac{\Gamma(a)\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z)\,, &\text{(A.16)}
\end{aligned}
$$

we can show that the above integrals satisfy the following relations

$$
\begin{aligned}
I_1(a,b,c;z) &= \frac{\sin\pi a}{\sin\pi(b+c)} I_1(b,a,c;1-z) - \frac{\sin\pi c}{\sin\pi(b+c)} I_2(b,a,c;1-z)\,, &\text{(A.17)}\\
I_2(a,b,c;z) &= -\frac{\sin\pi(a+b+c)}{\sin\pi(b+c)} I_1(b,a,c;1-z) - \frac{\sin\pi b}{\sin\pi(b+c)} I_2(b,a,c;1-z)\,. &\text{(A.18)}
\end{aligned}
$$

A formula of the hypergeometric function

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z) \tag{A.19}$$

is also used.

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